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STABILITY OF ROTATION OF A DEFORMABLE SPACECRAFT*

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Using Lur'e's approach to the description of non-small motions of a deformable system /1/, we derive by Kane's method /2/ the general equations of orbital motion of an elastic spacecraft filled with fluid. Generalizing the results of /3-5/ using the Thomson-Tate-Chetayev theorems, we obtain the conditions of asymptotic stability of rotation of a deformable spacecraft allowing for damping.

The method used for stability analysis is described in general in /3/. This method is employed to derive sufficient conditions of stability of uniform rotation of a thin elastic shell filled with fluid /6/.

1. The motion of a spacecraft relative to the inertial space may be characterized by large displacements and velocities, while the deformations are usually small. We introduce a locally fixed body system of coordinates $Ox_1x_2x_3$ so that for small deformations the relative displacements of the spacecraft particles is small. The relative position of the spacecraft particles in the undeformed state is determined by the radius-vector $\boldsymbol{\rho}(x_1, x_2, x_3)$, and their relative position in the deformed state by the radius-vector $\mathbf{r} = \boldsymbol{\rho} + \mathbf{u}\ (x_1, x_2, x_3, t)$. The motion of the body system of coordinates relative to the inertial space is determined by the translational velocity vector $\mathbf{v}_0$ of the pole O and the vector of the angular velocity of rotation $\boldsymbol{\omega}$ of this system about its pole. The vectors of absolute velocity and acceleration of a spacecraft particle are given by

$$\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{u}^{\cdot} \tag{1.1}$$
$$\mathbf{a} = \mathbf{v}_0^{\cdot} + \boldsymbol{\omega} \times \mathbf{v}_0 + \boldsymbol{\omega}^{\cdot} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{u}^{\cdot} + \mathbf{u}^{\cdot\cdot}$$

The forces acting on a volume element dV of an elastic body are expressible in terms of the stress tensor Θ:

$$\mathbf{f}_e = \operatorname{div} \Theta\, dV \tag{1.2}$$

A similar internal force field is produced by the forces of normal pressure and shearing stress in the fluid. The mass gravitational forces acting on the mass element dm are given by the approximate formula

$$\mathbf{f}_g = \{\mathbf{g} + \Omega^2 [3g^{-2}\mathbf{g}(\mathbf{r}\cdot\mathbf{g}) - \mathbf{r}]\}\, dm \tag{1.3}$$

where g is the vector of local free fall acceleration, and Ω^2 is the orbital angular velocity.

Application of D'Alembert's principle to the mass element dm of a deformable body produces the equation

$$\mathbf{a}dm = \mathbf{f}_e + \mathbf{f}_g \tag{1.4}$$

which is also true for systems with non-holonomic constraints. It may be supplemented with the condition of continuity of the medium and with boundary conditions on the walls of the cavity and on the boundary of the elastic medium.

Various discretization techniques are applied in order to write the equations of motion of a system in a form convenient for integration. Solutions of boundary-value problems of elasticity theory and hydrodynamics are usually available only for simple constructions. We correspondingly define a complete system of functions φ_n, which approximate the free oscillation modes of the real construction. The relative displacements of the construction particles are represented by a quadratic series expansion /1/

$$\mathbf{u} = \varphi_n(x_1, x_2, x_3)\, q_n(t) + {}^1/_2\, \varphi_{nm}(x_1, x_2, x_3)\, q_n(t)\, q_m(t) \tag{1.5}$$

where the functions φ_{nm} are expressible in some way in terms of φ_n.

Kane /2/ made an attempt to combine the physical lucidity of D'Alembert's principle with the procedure of automatic elimination of constraint reactions in Lagrange equations of the second kind. Kane's method of constructing the equations of motion is quite simple and has definite advantages for non-holonomic systems, since it does not require the introduction of undetermined Lagrange multipliers. In the linear case, it is identical with the Bubnov-

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Galerkin method.

Assume that the motion of the system is described by the quasicoordinates π_k, which determine the translational and rotational motion of the entire system, as well as its deformations. The vector which appears as the coefficient of the quasivelocity $\pi_k^{\cdot}$ is called the partial velocity of the quasicoordinate π_k. It may be determined from (1.1), (1.5) as the partial derivative of v with respect to the quasivelocity $\pi_k^{\cdot}$. Then the left- and the right-hand sides of Eq.(1.4) are scalar-multiplied by the corresponding velocity and integrated over the entire volume V occupied by the system particles. Combining the three equations for the quasicoordinates of translational motion, we obtain a vector equation of forces, and for the quasicoordinates of rotational motion we obtain a vector equation of momenta. For all the other coordinates, we obtain an infinite system of ordinary differential equations. If we introduce the apparent acceleration vector j, then the equation of motion of the deformable system takes the form

$$\begin{aligned}
&m\mathbf{j}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{L})+\boldsymbol{\omega}^{\cdot}\times\mathbf{L}+2\boldsymbol{\omega}\times\mathbf{L}^{\cdot}+\mathbf{L}^{\cdot\cdot}=\mathbf{P}+\\
&\quad\Omega^2[3g^{-2}\mathbf{g}(\mathbf{L}\cdot\mathbf{g})-\mathbf{L}]\\
&(J\cdot\boldsymbol{\omega})^{\cdot}+\boldsymbol{\omega}\times(J\cdot\boldsymbol{\omega})+\mathbf{L}\times\mathbf{j}+\boldsymbol{\omega}\times\mathbf{G}^*+\mathbf{G}^{*\cdot}=\\
&\quad\mathbf{M}+3\Omega^2g^{-2}[\mathbf{g}\times(J\cdot\mathbf{g})]\\
&(\mu_{nm}q_m^{\cdot})^{\cdot}+c_{nm}q_m+\frac{\partial\mathbf{L}}{\partial q_n}\cdot\mathbf{j}-{}^1/_2\,\boldsymbol{\omega}\cdot\frac{\partial J}{\partial q_n}\cdot\boldsymbol{\omega}+\\
&\quad\left(\frac{\partial\mathbf{G}^*}{\partial q_n^{\cdot}}\cdot\boldsymbol{\omega}\right)^{\cdot}-\boldsymbol{\omega}\cdot\frac{\partial\mathbf{G}^*}{\partial q_n}-{}^1/_2\,\frac{\partial\mu_{nm}}{\partial q_n}q_n^{\cdot}q_m^{\cdot}=\\
&\quad Q_n+{}^1/_2\,\Omega^2[3g^{-2}\mathbf{g}\cdot(J\cdot\mathbf{g})-\operatorname{tr}J]\\
&(\mathbf{j}=\mathbf{v}_0^{\cdot}+\boldsymbol{\omega}\times\mathbf{v}_0-\mathbf{g};\quad n,m=1,2,\ldots)
\end{aligned}\tag{1.6}$$

Here we use the following notation for the expansions in the coordinates q_n of the expressions for the static moment $\mathbf{L}$, the inertia tensor J, the angular momentum vector $\mathbf{G}^*$ and the kinetic energy T^* of relative motion:

$$\begin{aligned}
&\mathbf{L}=\int\mathbf{r}\,dm=\mathbf{L}_0+\mathbf{L}_{0n}q_n+{}^1/_2\,\mathbf{L}_{nm}q_nq_m\\
&J=\int[(\mathbf{r}\cdot\mathbf{r})E-\mathbf{r}:\mathbf{r}]\,dm=J_0+(J_{0n}+J_{0n}^T)q_n+J_{nm}q_nq_m\\
&\mathbf{G}^*=\int\mathbf{r}\times\mathbf{u}^{\cdot}\,dm=(\mathbf{G}_{0n}+\mathbf{G}_{nm}q_m)q_n^{\cdot}\\
&T^*={}^1/_2\int\mathbf{u}^{\cdot}\cdot\mathbf{u}^{\cdot}\,dm={}^1/_2\,\mu_{nm}q_n^{\cdot}q_m^{\cdot}=\\
&\quad{}^1/_2\,(a_{nm}+\mu_{nmk}q_k+\mu_{nmkl}q_kq_l)q_n^{\cdot}q_m^{\cdot}
\end{aligned}\tag{1.7}$$

The tensor summation convention over dummy indices is adopted, and the dyadic vector product is denoted by a colon:. The external forces, momenta, and generalized forces are denoted by $\mathbf{P}$, $\mathbf{M}$, and Q_n, respectively. The coefficients c_{nm} are determined by the rigidity of the elastic construction, and m is the mass of the entire deformable system.

2. We will investigate the stability of stationary rotation of a free deformable system. As the Lyapunov function of a holonomic system, we may use the Hamiltonian, defined equal to the sum of the kinetic energy and the changed potential energy. As shown in /3/, the perturbed motion asymptotically tends to stationary rotation: if the changed potential energy has an isolated minimum, dissipation is complete. Stability of the stationary motion of a system without dissipation, i.e., gyroscopic stabilization, may be achieved even if there is no minimum, provided the number of negative Poincaré stability coefficients is even.

Dissipative forces in a mechanical system are taken into account by the right-hand sides of Eqs.(1.6). If dissipation is complete, then the system Hamiltonian is a strictly decreasing function. However, for a free construction, the dissipative function determined by internal dissipation of energy by deformations is only semidefinite.

It was shown in /4/ by using the first integrals of the angular momentum that a free mechanical system experiences complete (propagating) damping in all the non-cyclic coordinates, since there are no perturbed paths in the neighbourhood of stationary rotation which are identically free energy dissipation. Therefore, by Krasovskii's theorem, the stability of stationary rotation of a mechanical system with damping can be inferred from the positive definiteness of the changed potential energy.

We will represent any vector as the product of the three-dimensional row of the unit vectors of the axes $Ox_1x_2x_3$ by the column matrix of its three components along the corresponding axes. We replace the vectors $\mathbf{r}, \boldsymbol{\omega}, \mathbf{j}, \mathbf{P}, \mathbf{M}, \mathbf{L}, \mathbf{G}^*$, etc., with the three-dimensional column matrices r, ω, j, P, M, L, G^*, and the tensors J with square matrices J. We denote the infinite-dimensional columns of the coordinates q_n, L_{0n}, Q_n, G_{0n}, J_{0n} by q, L_1, Q, G_1, J_1, and the infinite-dimensional square matrices with the elements a_{nm}, c_{nm}, L_{nm}, G_{nm}, J_{nm} by A, C, L_2, G_2, J_2, respectively. Ignoring the non-linear terms in small deformations, we rewrite the system of Eqs.(1.6) in matrix form as

$$
\begin{aligned}
& mj - L^b\omega^{\cdot} + \omega^b(\omega^b L) + 2\omega L^{\cdot} + L^{\cdot\cdot} = P \\
& (J\omega)^{\cdot} + L^b j + \omega^b(J\omega) + \omega^b G^* + G^{*\cdot} = M \\
& Aq^{\cdot\cdot} + Cq + j^T\frac{\partial L}{\partial q} - {}^1/_2\,\omega^T\frac{\partial J}{\partial q}\omega + \omega^{\cdot T}\frac{\partial G^*}{\partial q^{\cdot}} + 2\omega^T\frac{\partial G^*}{\partial q} = Q
\end{aligned}
\tag{2.1}
$$

The superscript b denotes skew-symmetric matrices formed by the vector product rule.
In the case of stationary rotation without external forces (other than dissipative forces), this system may be linearized in the perturbations of the generalized coordinates. Select the body system of coordinates $Ox_1x_2x_3$ so that its origin coincides with the centre of mass of the undeformed body, and the axes are parallel to the principal axes of inertia. The angular velocity of stationary rotation ω_0 is assumed to be a finite quantity directed along the Ox_1 axis; all other coordinates are assumed small. We introduce special notation for the components of the following matrices along the three axes:

$$
\begin{aligned}
& L_1 = \begin{Vmatrix} l_1 \\ l_2 \\ l_3 \end{Vmatrix}, \quad G_1 = \begin{Vmatrix} g_1 \\ g_2 \\ g_3 \end{Vmatrix}, \quad G_2 = \begin{Vmatrix} g_{01} \\ g_{02} \\ g_{03} \end{Vmatrix}, \quad \omega = \begin{Vmatrix} \omega_0 + \omega_1^* \\ \omega_2 \\ \omega_3 \end{Vmatrix} \\
& J_1 = \| d_{kl} \|, \quad J = \| d_{0kl} \|
\end{aligned}
\tag{2.2}
$$

Eliminating the apparent acceleration j and the cyclic coordinate ω_1^*, we obtain a system of equations of perturbed motion in the variable $z^T = \| \omega_2\omega_3 q \|$ in matrix form,

$$
M_0 z^{\cdot\cdot} + G_0 z^{\cdot} + K_0 z = -F_0 z^{\cdot} \tag{2.3}
$$

The square matrices are determined by the scalar coefficients of the equations of motion (1.6) or (2.1),

$$
\begin{aligned}
& M_0 = \begin{Vmatrix} 0 & 0 & g_2^T \\ 0 & 0 & g_3^T \\ 0 & 0 & A_* \end{Vmatrix}, \quad K_0 = \omega_0^4 \begin{Vmatrix} 0 & J_{11} - J_{33} & -d_{13}^T \\ J_{22} - J_{11} & 0 & d_{12}^T \\ -d_{12} & -d_{13} & C_* \end{Vmatrix} \\
& G_0 = \omega_0 \begin{Vmatrix} J_{22} & 0 & d_{12}^T - g_3^T \\ 0 & J_{33} & d_{13}^T + g_2^T \\ g_2 & g_3 & G_* \end{Vmatrix}, \quad F_0 = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{Vmatrix}
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
& A_* = A - \frac{L_1 L_1^T}{m} - \frac{g_1 g_1^T}{J_{11}} \\
& C_* = \frac{C}{\omega_0^2} - d_{011} + \frac{l_2 l_2^T + l_3 l_3^T}{m} + \frac{d_{11} d_{11}^T}{J_{11}} \\
& G_* = g_{01}^T - g_{01} + 2\frac{l_3 l_2^T - l_2 l_3^T}{m} + \frac{g_1 d_{11}^T - d_{11} g_1^T}{J_{11}}
\end{aligned}
\tag{2.5}
$$

The system of Eqs.(2.3) is expressed in quasicoordinates, and its matrices do not possess the required symmetry. If we pass from quasicoordinates to generalized coordinates θ_k (Cardan angles) using the formulas

$$
\omega_1 = \omega_0 + \theta_1^{\cdot}, \ \omega_2 = \theta_2^{\cdot} - \omega_0\theta_3, \ \omega_3 = \theta_3^{\cdot} + \omega_0\theta_2 \tag{2.6}
$$

we obtain a linearized system of Lagrange equations of the second kind in the new variable $y^T = \| \theta_2\theta_3 q \|$:

$$
M_1 y^{\cdot\cdot} + G_1 y^{\cdot} + F_1 y^{\cdot} + Ky = 0 \tag{2.7}
$$

$$
\begin{aligned}
& M_1 = \begin{Vmatrix} J_{22} & 0 & g_2^T \\ 0 & J_{33} & g_3^T \\ g_2 & g_3 & A_* \end{Vmatrix}, \quad K_1 = \omega_0^6 \begin{Vmatrix} J_{11} - J_{33} & 0 & -d_{13}^T \\ 0 & J_{11} - J_{22} & d_{12}^T \\ -d_{13} & d_{12} & C_* \end{Vmatrix} \\
& G_1 = \omega_0^3 \begin{Vmatrix} 0 & J_{11} - J_{22} - J_{33} & d_{12}^T - g_3^T \\ J_{33} + J_{22} - J_{11} & 0 & d_{13}^T + g_2^T \\ -(d_{12} - g_{13}) & -(d_{13} + g_2) & G_* \end{Vmatrix}, \quad F_1 = F_0
\end{aligned}
\tag{2.8}
$$

By the Thomson-Tate-Chetayev theorems, if M_1 is a symmetric positive definite matrix, G_1 a skew-symmetric matrix, F_1 a symmetric non-negative matrix without the complete dissipation property, and K_1 a symmetric matrix, then the zero solution of the system (2.7) is asymptotically stable when K_1 is positive define and unstable when K_1 has negative eigenvalues, independently of G_1 and F_1. The matrix K_1 is the Hessian of the changed potential energy. Therefore, the condition of positive definiteness of K_1 is identical with the condition for a minimum of the changed potential energy. Completeness of damping in the linear system (2.7) with semi-definite matrix F_1 can be established using the controllability property of this system. For damping to be complete, it is necessary and sufficient that the controllability matrix is of full rank.

All the theorems on the stability of the zero solution of the linearized system (2.7) are automatically extended to the zero solution of system (2.3), since the variables y and z are linked by a non-singular linear transformation (2.6). From formulas (2.4) and (2.8) we see that if the first and the second column in the matrix K_0 are interchanged, then this matrix will be identical with K_1 apart from a constant factor ω_0^2. Therefore, the determinant of the matrix K_1 is equal to the determinant of the matrix K_0 apart from a constant factor ω_0^2.

Asymptotic stability requires positive definiteness of the matrix K_1, and by the Sylvester criterion this implies that all the main diagonal minors should be positive. The first and the second upper minors provide the necessary condition of stability of rotation of the deformable system around its principal axis of the maximum moment of inertia in the unperturbed state. All the lower main minors, defined by the matrix C_*, are positive, because an elastic system with positive-definite potential deformation energy is conservative.

The matrix K_1 can be partitioned into blocks

$$K_1 = \begin{Vmatrix} \Delta & D^T \\ D & C_* \end{Vmatrix}, \quad \Delta = \begin{Vmatrix} J_{11} - J_{33} & 0 \\ 0 & J_{11} - J_{22} \end{Vmatrix}, \quad D = \| -d_{13} d_{12} \| \tag{2.9}$$

Since the matrix C_* is square and non-singular, the inverse C_*^{-1} exists. Therefore, the computation of the determinant of the matrix K_1 can be reduced to the computation of lower-order determinants:

$$\det K_1 = \det \begin{Vmatrix} \Delta - D^T C_*^{-1} D & 0 \\ D & C_* \end{Vmatrix} = \det C_* \det (\Delta - D^T C_*^{-1} D) \tag{2.10}$$

The first factor $\det C_*$ is positive. Therefore, the sign of $\det K_1$ is identical with the sign of the second factor in (2.10). Similarly, the two smaller main minors of the matrix K_1 are representable as a product of two factors, the first of which is positive.

If the potential deformation energy is expressed in terms of the principal modes of the rotating construction, then the matrices C_* and C_*^{-1} are diagonal. In the case, using the notation (2.2), we have the following identities:

$$d_{12}{}^T C_*^{-1} d_{12} = \frac{(d_{12}{}^n)^2}{c_{nn}} = t_2, \quad d_{13}{}^T C_*^{-1} d_{13} = \frac{(d_{13}{}^n)^2}{c_{nn}} = t_3 \tag{2.11}$$

Putting

$$\Delta_2 = \frac{J_{11} - J_{22}}{t_2}, \quad \Delta_3 = \frac{J_{11} - J_{33}}{t_3}, \quad t = \frac{1}{t_2 t_3} \left(\frac{d_{12}{}^n d_{13}{}^n}{c_{nn}} \right)^2 \tag{2.12}$$

we obtain the conditon for stability in the form

$$\Delta_2 > 0, \ \Delta_3 > 0, \ \Delta_2 > 1, \ \Delta_3 > 1 \ (\Delta_2 - 1)(\Delta_3 - 1) - t > 0 \tag{2.13}$$

By the Cauchy inequality, the non-negative quantity t is always less than or equal to 1. The equality sign applies for $d_{12} = d_{13}$.

In Fig.1, in the plane of the parameters Δ_2, Δ_3, the region of asymptotic stability (horizontal shading) lies above the curve *1* and corresponds to the last condition in (2.13). The first two inequalities, defining the first quadrant, and the next two inequalities, defining the area inside this quadrant, are a priori satisfied in the shaded region above curve *1*. The explicit form of series (2.11) and (2.12), which occurs in the sufficient condition of stability (2.13), provides a direct estimate of the error associated with the high-frequency cutoff of the elastic oscillations in the initial system of equations of motion. When the higher frequencies are cut off, we obtain a somewhat larger stability domain due to the errors in computing t_2 and t_3.

On the other hand, applying the Routh-Hurwitz criterion to the initial system (2.3), we find that a necessary condition of asymptotic stability is the positivity of the free term in the characteristic equation, which is equal to $\det K_0 \equiv \omega^{-2} \det K_1$. In order to ensure the sufficient conditions of asymptotic stability, we also require the positivity of all the main minors of the Hurwitz determinant, which are quite cumbersome already with three degrees of freedom. We have shown above that in the case of free mechanical systems with propagating damping and a Hamiltonian that does not depend explicitly on time, the condition $\det K_1 > 0$, and therefore the condition $\det K_0 > 0$ is necessary and sufficient for the asymptotic stability of the stationary rotation of a deformable system.

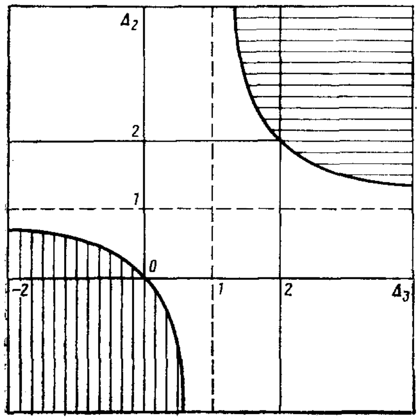

Fig.1

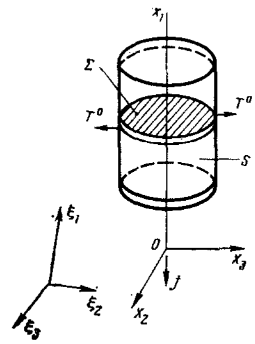

Fig.2

This result has been proved directly by determining the signs of the coefficients of the leading terms in the Sturm series polynomials for the special case of an axisymmetric construction with four radial booms, allowing for low-frequency elastic oscillations /7/. The boundaries of the stability region (curve *1* in Fig.1) have the following physical meaning: the rotation of the deformable system is stable relative to the axis of the maximum moment of inertia only when this moment exceeds the other principal moments of inertia by some quantity dependent on the elastic characteristics of the system. This quantity vanishes if the rigidity C of the system increases to infinity, if the velocity of rotation ω_0 decreases, and, finally, if the inertial coupling coefficients d_{12}, d_{13} are small, e.g., due to the small mass and size of the deformable elements.

If there are no substantial energy losses in the elastic body, then, alongside the region of asymptotic stability, there may also exist a region of temporary gyroscopic stabilization, which unlike secular asymptotic stability breaks down in an infinite time interval. A necessary condition of gyroscopic stabilization is that the matrix of conservative forces K_1 has even negative roots. In Fig.1, the region of gyroscopic stabilization is shown by vertical shading below curve *2*. In a real mechanical system, even small dissipative forces will destroy this region over time. If the system is non-conservative, then domains of strong instability may form inside the region of gyroscopic stability. An example of such domains for the eddy motion of a fluid completely filling a rotating cylindrical vessel is given in /8/.

3. As an example (Fig.2), consider the stability of rotation of a body with an axisymmetric cavity filled with an ideal incompressible fluid. The free surface of the fluid Σ supports a circular plate with bending rigidity D° and median stress T°. The longitudinal axis of the body Ox_1 is the axis of symmetry of the cavity along which the load j_1 is applied. Potential fluid flow is assumed. In this case, the fluid velocity potential Φ and the deformation u of the plate satisfy the following system of boundary-value problems:

$$\begin{gathered}\nabla^2\Phi = 0,\ \partial\Phi/\partial\nu\,|_S = (r\times\nu)\cdot\omega,\ \partial\Phi/\partial\nu\,|_\Sigma = du/dt + (r\times\nu)\cdot\omega \\ \rho^\circ\delta^\circ\partial^2u/\partial t^2 + D^\circ\nabla^2\nabla^2u - T^\circ\nabla^2u = -\rho^1\,[\partial\Phi/\partial t + j_1x_1 + {}^1/_2\,(\nabla\Phi)^2 - \\ (\omega\times r)\cdot\nabla\Phi + c\,(t)] \\ u\,|_{\Gamma_*} = 0,\ \partial u/\partial n\,|_{\Gamma_*} = 0,\ u\,|_\Sigma < \infty,\ \nabla u\,|_\Sigma < \infty\end{gathered} \tag{3.1}$$

Here to be specific we assume rigid clamping conditions: ν is the unit vector of the outer normal to the surface of the fluid $S + \Sigma$, where S is the wetted surface of the cavity; n is the unit vector of the outer normal to the plate contour Γ_*, ρ°, δ° are the density and the thickness of the plate, and ρ^1 is the fluid density.

Investigation of the free oscillations of a fluid in a mass force field leads to the boundary-value problem

$$\nabla^2\varphi = 0,\ \partial\varphi/\partial\nu\,|_S = 0,\ \partial\varphi/\partial\nu\,|_\Sigma = \varkappa\varphi \tag{3.2}$$

This problem has an infinite discrete spectrum of eigenvalues $\varkappa_i$ and the corresponding complete system of functions φ_i orthogonal on Σ. Series-expanding in these functions the solution of the homogeneous linear problem (3.1) for the case of a stationary cavity, we obtain the free hydroelastic modes of the plate in the form /9/

$$v_n = \sum_{i=1}^{\infty} b_{in} \frac{\partial \varphi_i}{\partial \nu}, \quad b_{in} = \frac{C_2 E_{1i}{}^\circ - C_1 E_{2i}{}^\circ}{(\omega_n{}^2 - \alpha_{in})(C_2 B_1 - C_1 B_2)} \alpha_{in} \tag{3.3}$$

$$C_l = \left.\frac{\partial v_l{}^\circ}{\partial n}\right|_{\Gamma_*}, \quad E_{li}{}^\circ = \frac{1}{N_i{}^2} \int v_l{}^\circ \frac{\partial \varphi_i}{\partial \nu} ds$$

$$N_i{}^2 = \int\limits_\Sigma \left(\frac{\partial \varphi_i}{\partial \nu}\right)^2 ds, \quad B_l = v_l{}^\circ |_{\Gamma_*} \quad (l = 1, 2)$$

$$\alpha_{in} = [D^\circ \nabla^2 \nabla^2 \varphi_i - T^\circ \nabla^2 \varphi_i + (\rho^1 j_1 - \rho^\circ \delta^\circ \omega_n{}^2) \varphi_i] / \rho^1 \varphi_i$$

Here $v_l{}^\circ$ are two independent solutions of the homogeneous equation of the oscillation of the plate, and the oscillation frequency ω_n is the k-th root of the characteristic equation

$$1 - \omega_n{}^2 \sum_{i=1}^{\infty} \frac{B_i{}^* b_{in}}{\alpha_{in}} + \frac{\omega_n{}^2}{C_2 B_1 - C_1 B_2} [(B_2 - \omega_n{}^2 \Sigma_{B2}) \Sigma_{C1} - (B_1 - \omega_n{}^2 \Sigma_{B1}) \Sigma_{C2}] = 0 \tag{3.4}$$

$$\sum_{Al} = \sum_{i=1}^{\infty} \frac{A_i{}^* E_{li}{}^\circ}{\omega_n{}^2 - \alpha_{in}} \quad (A = B, C; \quad l = 1, 2) \tag{3.5}$$

$$B_i{}^* = \varkappa_i \varphi_i |_{\Gamma_*}, \ C_i{}^* = \varkappa_i \partial \varphi_i \mid \partial n |_{\Gamma_*}$$

Using the eigenfunctions v_n, we can also construct a solution of the system of partial differential Eqs.(3.1) for the case of a moving cavity.

$$u = v_n (x_2, x_3) p_n (t) \tag{3.6}$$
$$\Phi = (\Omega_0 + \Omega_n p_n + \Omega_{nm} p_n p_m) \cdot \omega + \varphi_i b_{in} p_n{}^\cdot$$

where the components of the vector Ω_0 are the Zhukovskii potentials, and the functions Ω_n, Ω_{nm} are determined by the non-linear terms of system (3.1). The corresponding boundary-value problems with their solutions can be found in /10/. Assuming that the fluid mass is much greater than the mass of the plate, we obtain expressions (1.8) in the form

$$L_{0n} = b_{in} \lambda_i, \quad G_{0n} = b_{in} \lambda_{0i}, \quad G_{nm} = b_{in} b_{jm} \lambda_{ij} \tag{3.7}$$
$$J_{0n} = b_{in} J_i, \quad J_{nm} = b_{in} b_{jm} J_{ij}, \quad a_{nm} = b_{in} b_{jm} \mu_i \delta_{ij}$$

where b_{in} are from (3.3) and the remaining hydrodynamic characteristics of the cavity were determined in /10/.

In the case of a cavity of revolution, the oscillations of the fluid and the plate fall into two groups: a group of oscillations symmetric about the plane Ox_1x_2 and a group of oscillations symmetric about the plane Ox_1x_3. Let the first N modes v_n and the corresponding $\varphi_i^{(p)}$ in Eqs.(3.6) be even functions relative to the axis Ox_2 and the following modes v_{N+n} with their corresponding $\varphi_i^{(q)}$ be even functions relative to the axis Ox_3. Retaining our notation for the first group of generalized coordinates p_n $(n \leqslant N)$, we replace the coordinates p_{N+n} in the second group by q_n $(n < N)$. The non-zero coefficients in the formulas (3.7) are the following (allowing for the symmetry properties):

$$\lambda_i = (\lambda_i{}^p)_3 = (\lambda_i{}^q)_2, \quad \mu_i = (\lambda_{ii}^{pq})_1 = -(\lambda_{ii}^{qp})_1 = (J_{ii}^{pp})_{11} = (J_{ii}^{qq})_{11} \tag{3.8}$$
$$\lambda_{0i} = -(\lambda_{0i}{}^p)_2 = (\lambda_{0i}{}^q)_3 = -(J_i{}^p)_{13} = -(J_i{}^p)_{31} = -(J_i{}^q)_{12} = -(J_i{}^q)_{21}$$

The non-zero elements of the matrices (2.2), by formulas (3.7) and (3.8), take the form

$$l_2^{N+n} = l_3{}^n = b_{in} \lambda_i \tag{3.9}$$
$$g_3^{N+n} = -g_2{}^n = -d_{31}{}^n = -d_{12}^{N+n} = -d_{21}^{N+n} = b_{in} \lambda_{0i}$$
$$a_{nm} = g_{01}^{n,\, N+m} = -g_{01}^{N+m,\, n} = d_{011}^{n,\, m} = d_{011}^{N+n,\, N+m} = b_{in} b_{im} \mu_i$$

Substituting these formulas into Eqs.(2.3) and using the orthogonality conditions, we see that the equations of the momenta about the longitudinal axis and the equation of forces in the direction of this axis degenerate and become decoupled. If the inertia tensor of the body is also axially symmetric, then, using the notation

$$J_{22} = J_{33} = J, \ r_n = p_n - i q_n, \ \omega = \omega_2 + i \omega_3, \ j = j_3 - i j_2 \tag{3.10}$$

we can rewrite the system of Eqs.(2.3) in complex form,

$$J \omega^\cdot - i (J_{11} - J) \omega_0 \omega - b_{in} \lambda_{0i} (r_n{}^{\cdot\cdot} + 2 i \omega_0 r_n{}^\cdot - \omega_0{}^2 r_n) = 0 \tag{3.11}$$
$$b_{im} b_{in} \mu_i \{(1 - \chi)(r_m{}^{\cdot\cdot} + 2 i \omega_0 r_m{}^\cdot) + [\omega_m{}^2 - (1 - \chi) \omega_0{}^2] r_m\} - b_{in} \lambda_{0i} (\omega^\cdot + i \omega_0 \omega) = 0$$

$$\chi = b_{jm} b_{in} \lambda_j \lambda_i / (m b_{im} b_{in} \mu_i)$$

Since $t \equiv 0$ in formulas (2.12), the necessary and sufficient condition for asymptotic stability (2.13) is simplified in this case. If we allow for the orthogonality of the elastic oscillation modes of the plate and ignore the non-diagonal temrs in (3.11), then the following inequality is sufficient for stability of rotation of a body with a fluid and a plate:

$$J_{11} - J > \frac{(b_{in} \lambda_{0i})^2}{b_{in}^2 \mu_i [\omega_n^2 - (1 - \chi) \omega_0^2]} \tag{3.12}$$

The coefficients λ_{0i}, μ_i, λ_i are determined by the solutions of the linear boundary-value problem (3.2), and the coefficients b_{in}, ω_n^2 are given by formulas (3.3), (3.4). We assume that the angular velocity of rotation ω_0 is less than any elastic oscillation frequency of the plate ω_n. For the case of a cylindrical cavity, numerical values of the frequencies ω_n and of the other coefficients in (3.12) were obtained in /9/ for various bending rigidities, stresses, and loads.

REFERENCES

1. LUR'YE A.I., Analytical Mechanics, Moscow, Fizmatgiz, 1961.
2. KANE T.R. and LEVINSON D.A., Formulation of equations of motion for complex spacecraft, J. Guidance and Control, 3, 2, 1980.
3. RUMYANTSYEV V.V., On the motion and stability of an elastic body with a cavity containing a fluid, PMM, 33, 6, 1969.
4. PRINGLE R., On the stability of a body with connected moving parts, AIAA J. 4, 8, 1966.
5. SAMIN J.C. and WILLEMS P.V., On the attitude dynamics of spinning deformable systems, AIAA J, 13, 6, 1975.
6. RUBANOVSKII V.N., Stability of stationary motions of a solid with an elastic shell partially filled with fluid, PMM, 46, 4, 1982.
7. DOKUCHAYEV L.V., Constructing the regions of stable rotation of a spacecraft with elastic booms, Kosmich. Issledovaniya, 7, 4, 1969.
8. DOKUCHAYEV L.V. and RVALOV R.V., On the stability of stationary rotation of a solid with a cavity containing a fluid, Izv. Akad. Nauk SSSR, MTT, 2, 1973.
9. DOKUCHAYEV L.V., On the oscillations of a reservoir with a fluid supporting a membrane on its free surface, Stroit. Mekhanika i Raschet Sooruzhenii, 1, 1972.
10. NARIMANOV G.S., DOKUCHAYEV L.V. and LUKOVSKII I.A., Non-linear Dynamics of a Spacecraft with Fluid, Moscow, Mashinostroenie, 1977.

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